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Planck scattering beyond the eikonal approximation in the functional approach

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Abstract. In the framework of functional integration the non-leading terms of the leading eikonal behavior of the Planck energy scattering amplitude are calculated by the straight-line path approximation. We show that the allowance for the first-order correction terms leads to the appearance of the retardation effect. The singular character of the correction terms at short distances is also noted, and they may ultimately lead to the appearance of non-eikonal contributions to the scattering amplitudes.

1 Introduction

The asymptotical behavior of the scattering amplitude at high energy is one of the central problems of elementary particle physics. The standard method of quantum field theory is expected to entail that the calculations based on perturbation theory are suitable when the energy of individual particles is not rather high and the effective coupling constant is not large. When the energy is increased the effective coupling constant also increases, so that the corrections calculated by perturbation theory play a crucial role. Gravitational scattering occurs at Planck energy $s^{1/2} = 2E \geq M_{\mathrm{PL}}$, where s is the square of the center of mass energy, M_{PL} is the Planck mass, and small angles are characterized by the effective coupling constant $\alpha_G = Gs/\hbar \geq 1$ which makes any simple perturbative expansion unwarranted. Comparison of the results of the different approaches [1–3,7,8] proposed for this problem has shown that they all coincide in the leading order approximation, which has a semiclassical effective metric interpretation, while most of them fail in providing the non-leading terms under which new classical and quantum effects are hiding [2,3].

The aim of the present paper is to continue the determination of the non-leading terms to the Planck energy scattering by a functional approach proposed for constructing a scattering amplitude in our previous works [9, 10]. Using the straight-line path approximation we have shown that in the limit of asymptotically high $s \gg M_{\mathrm{PL}}^2 \gg t$, at fixed momentum transfers t the lowest order eikonal expansion of the exact two-particle Green function on the mass shell gives the leading behavior of the Planck energy scattering amplitude, which agrees with the results found by all others [1–3,7,8]. The main advantage of the proposed approach is the possibility of performing calculations in a compact form and obtaining the sum of the considered diagrams immediately in a closed form.

The outline of this paper is as follows. In the second section using the example of the scalar model $L_{\mathrm{int}} = g\varphi^2\phi$, which allows one to make the exposition having most clarity and being most descriptive, and also less tedious calculations being involved, by means of the functional integration, we briefly demonstrate the conclusion of the leading behavior [9–14,28,16,17] and explain the important steps in calculating the non-leading terms to the high-energy scattering amplitude [28]. This section can be divided into three parts. In the first one the quantum Green function of two particles is obtained in the form of the functional integral. In the second part by a transition to the mass shell of the external two-particle Green function we obtain a closed representation for the two-particle scattering amplitude which is also expressed in the form of functional integrals. In the last of this section the straight-line path approximation and its generalization are discussed for calculating the non-leading terms to high-energy scattering amplitudes. Based on the exact expression of the single-particle Green function in the gravitational field $g_{\mu\nu}(x)$ obtained in [9], the results discussed in the second section will be generalized in the third section to the case of scalar "nucleons" of the field $\varphi(x)$ interacting with a gravitational field. Finally, in the fourth section we draw our conclusions.

2 Corrections to the eikonal equations in the scalar model

In the construction of a scattering amplitude we use a reduction formula which relates an element of the S matrix to the vacuum expectation of the chronological product of the field operators. For the two-particle amplitude, this formula has the form

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$$\mathrm{i}(2\pi)^4\delta^4(p_1+p_2-q_1-q_2)T(p_1,p_2;q_1,q_2)$$
$$=\mathrm{i}^4\int\prod_{k=1}^{2}\mathrm{d}x_k\mathrm{d}y_k\overrightarrow{K}^m_{x_1}\overrightarrow{K}^m_{x_2}$$
$$\times\langle 0|T(\varphi(x_1)\varphi(x_2)\varphi(y_1)\varphi(y_2))|0\rangle\overleftarrow{K}^m_{y_1}\overleftarrow{K}^m_{y_2}, \quad (2.1)$$

where p_1, p_2 and q_1, q_2 are the moments of the particles of the field $\varphi(x)$ before and after scattering, respectively.

Ignoring the vacuum polarization effects the two-nucleon Green function on the right-hand side of (2.1) can be represented in the form

$$G(x_1,x_2;y_1,y_2)=\langle 0|T(\varphi(x_1)\varphi(x_2)\varphi(y_1)\varphi(y_2))|0\rangle$$
$$=\exp\left\{\frac{\mathrm{i}}{2}\int D\frac{\delta^2}{\delta\phi^2}\right\}\Big[G(x_1,y_1|\phi)G(x_2,y_2|\phi)$$
$$+G(x_1,y_2|\phi)G(x_2,y_1|\phi)\Big]\Big|_{\phi=0}, \quad (2.2)$$

where

$$\exp\left\{\frac{\mathrm{i}}{2}\int D\frac{\delta^2}{\delta\phi^2}\right\}$$
$$=\exp\left\{\frac{\mathrm{i}}{2}\int\mathrm{d}^4z_1\mathrm{d}^4z_2D(z_1-z_2)\frac{\delta^2}{\delta\phi(z_1)\delta\phi(z_2)}\right\}, \quad (2.3)$$

and $G(x,y|\phi)$ is the Green function of the nucleon $\varphi(x)$ in a given external field $\phi(x)$.The nucleon Green function $G(x,y|\phi)$ satisfies the equation

$$[\Box+m^2-g\phi(x)]G(x,y|\phi)=\delta^4(x-y), \quad (2.4)$$

whose formal solution can be written in the form of a Feynman path integral:

$$G(x,y|\phi)=\mathrm{i}\int_0^\infty\mathrm{e}^{-\mathrm{i}m^2\tau}\mathrm{d}\tau\int[\delta^4\nu]_0^\tau \quad (2.5)$$
$$\times\exp\left\{\mathrm{i}g\int\mathrm{d}zJ(z)\phi(z)\right\}\delta^4\left(x-y+2\int_0^\tau\nu(\eta)\mathrm{d}\eta\right),$$

where $J(z)$ is the classical current of the nucleon[1]:

$$J(z)=\int_0^\tau\mathrm{d}\eta\delta^4\left(z-x+2\int_0^\tau\nu(\xi)\mathrm{d}\xi\right), \quad (2.6)$$

$[\delta^4\nu_i]_{\tau_1}^{\tau^2}$ is a volume element of the functional space of the four-dimensional function $\nu(\eta)$ defined on the interval $\tau_1\le\eta\le\tau_2$,

$$[\delta^4\nu_i]_{\tau_1}^{\tau^2}=\frac{\delta^4\nu_i\exp[-\mathrm{i}\int_{\tau_1}^{\tau^2}\nu_\mu^2(\eta)\prod_\eta\mathrm{d}^4\eta}{\int\delta^4\nu_i\exp[-\mathrm{i}\int_{\tau_1}^{\tau^2}\nu_\mu^2(\eta)\prod_\eta\mathrm{d}^4\eta}.$$

Substituting (2.5) into (2.2) and performing the variational differentiation with respect to ϕ, we find that the Fourier transform of the two-nucleon Green function

$$G(p_1,p_2;q_1,q_2)$$
$$=\int\prod_{i=1}^{2}(\mathrm{d}^4x_i\mathrm{d}^4y_i\mathrm{e}^{\mathrm{i}(p_ix_i-q_iy_i)})G(x_1,x_2;y_1,y_2) \quad (2.7)$$

[1] In the scalar model $J(z)$ describes the spatial density of nucleon moving on a classical trajectory. However, in this case we call $J(z)$ a current

is given by the following expression:

$$G(p_1,p_2|q_1,q_2)$$
$$=\mathrm{i}^2\prod_{i=1}^{2}\left(\int_0^\infty\mathrm{d}\tau_i\mathrm{e}^{\mathrm{i}\tau_i(p_i^2-m^2)}\int[\delta^4\nu_i]_0^{\tau_i}\int\mathrm{d}x_i\mathrm{e}^{\mathrm{i}x_i(p_i-q_i)}\right)$$
$$\times\exp\left[-\frac{\mathrm{i}g^2}{2}\int D(J_1+J_2)^2\right]+(p_1\leftrightarrow p_2), \quad (2.8)$$

where we have introduced the abbreviated notation

$$\int J_iDJ_k=\int\int\mathrm{d}z_1\mathrm{d}z_2J_i(z_1)D(z_1-z_2)J_k(z_2). \quad (2.9)$$

Expanding the expression (2.8) with respect to the coupling constant g^2 and taking the functional integrals with respect to ν_i, which reduce to simple Gaussian quadratures if a Fourier transformation is made, we obtain the well-known series of perturbation theory for $G(p_1,p_2|q_1,q_2)$.

The elastic-scattering amplitude is related to the two-nucleon Green function by

$$\mathrm{i}(2\pi)^4\delta^4(p_1+p_2-q_1-q_2)T(p_1,p_2|q_1,q_2)^{\mathrm{scalar}}$$
$$=\lim_{p_i^2,q_i^2\to m^2}\left(\prod_{i=1,2}(p_i^2-m^2)(q_i-m^2)\right)G(p_1,p_2|q_1,q_2)$$
$$+(p_1\leftrightarrow p_2). \quad (2.10)$$

Substituting (2.5) into (2.2) and making a number of substitutions of the functional variables [9], we obtain a closed expression for the two-nucleon scattering amplitude in the form of functional integrals:

$$T(p_1,p_2;q_1,q_2)^{\mathrm{scalar}}=\frac{g^2}{(2\pi)^4}\int\mathrm{d}^4x\mathrm{e}^{\mathrm{i}(p_1-q_1)x}D(x)$$
$$\times\left(\prod_{i=1}^{2}\int[\delta^4\nu_i]_{-\infty}^{\infty}\exp\left\{\mathrm{i}\frac{g^2}{2}\sum_{i=1,2}\int(J_iDJ_i-\mathrm{i}\delta_im^2)\right\}\right)$$
$$\times\exp\int_0^1\mathrm{d}\lambda\exp\left(\mathrm{i}g^2\lambda\int J_1DJ_2\right)+(p_1\leftrightarrow p_2), \quad (2.11)$$

where the quantity $J_i(z,p_i,q_i|\nu_i)$ is a conserving transition current given by

$$J_i(z,p_i,q_i|\nu_i)$$
$$=\int_{-\infty}^{\infty}\mathrm{d}\xi\delta\left(z-x_i-a_i(\xi)+2\int_0^\xi\nu_i(\eta)\mathrm{d}\eta\right), \quad (2.12)$$
$$a_{1,2}(\xi)=p_{1,2}\theta(\xi)+q_{1,2}\theta(-\xi). \quad (2.13)$$

The scattering amplitude (2.11) is interpreted as the residue of the two-particle Green function (2.8) at the poles corresponding to the nucleon ends. A factor of the type $\exp\left(-(\mathrm{i}\kappa^2/2)\sum_{i=1,2}\int J_iDJ_i\right)$ of (2.11) takes into account the radiative corrections to the scattered nucleons, while $\exp\left(\mathrm{i}\kappa^2\lambda\mathrm{e}^{\mathrm{i}kx}\int J_1DJ_2\right)$ describes virtual-meson exchange among them. The integral with respect to $\mathrm{d}\lambda$

ensures the subtraction of the contribution of the freely propagating particles from the matrix element. The functional variables $\nu_1(\eta)$ and $\nu_2(\eta)$ formally introduced for obtaining the solution of the Green function describe the deviation of a particle trajectory from the straight-line paths. The functional with respect to $[\delta^4\nu_i]$ $(i = 1, 2)$ corresponds to the summation over all possible trajectories of the colliding particles. From the consideration of the integrals over ξ_1 and ξ_2 for $\exp\left(-(\mathrm{i}\kappa^2/2)\sum_{i=1,2}\int J_i D J_i\right)$ it is seen that the radiative correction result in divergent expressions of the type $\delta_i m^2 \times (A \to \infty)$. To regularize them, it is necessary to renormalize the mass, that is, to separate from $\exp\left(-(\mathrm{i}\kappa^2/2)\sum_{i=1,2}\int J_i D J_i\right)$ the terms $\delta_i m^2 \times (A \to \infty)$ $(i = 1, 2)$, after which we go over in (2.11) to the observed mass $m_{i\mathrm{R}}^2 = m_{i0}^2 + \delta_i m^2$. These problems have been discussed in detail in previous works [9,10,12, 18]; therefore we shall hereafter drop the radiation corrections terms $\exp\left(\mathrm{i}(g^2/2)\sum_{i=1,2}\int[J_i D J_i - \mathrm{i}\delta_i m^2]\right)$ as these contributions in our model can be factorized as a factor $R(t)$ that depends only on the square of the moment transfer. A similar factorization of the contributions of radiative corrections in quantum electrodynamics has also been obtained [19].

Ignoring the radiation corrections, the elastic-scattering amplitude of two scalar nucleons (2.11) can be represented in the following form:

$$T(p_1, p_2|q_1, q_2)^{\text{scalar}} = \frac{\mathrm{i}g^2}{(2\pi)^4}\int \mathrm{d}^4x \mathrm{e}^{-\mathrm{i}x(p_1-q_1)} D(x) \int_0^\lambda \mathrm{d}\lambda S_\lambda + (p_1 \leftrightarrow p_2), \tag{2.14}$$

where

$$S_\lambda = \int \prod_{i=1}^{2} [\delta^4\nu_i]_{-\infty}^{\infty} \exp\{\mathrm{i}g^2\lambda\Pi[\nu]\};$$
$$\Pi[\nu] = \int J_1 D J_2, \tag{2.15}$$

and the quantity $J_i(k, p_i, q_i|\nu_i)$ is a conserving transition given by

$$J_i(k, p_i, q_i|\nu_i) = \int_{-\infty}^{\infty} \mathrm{d}\xi \exp\left(2\mathrm{i}k\left[a_i(\xi) + \int_0^\xi \nu_i(\eta)\mathrm{d}\eta\right]\right). \tag{2.16}$$

Note that the expression (2.12) defines the scalar density of a classical point particle moving along the curvilinear path $x_i(s)$, which depends on the proper time $s = 2m\xi$ and satisfies the equation

$$m\mathrm{d}x_i(s)/\mathrm{d}s = p_i\theta(\xi) + q_i\theta(-\xi) + \nu_i(\xi) \tag{2.17}$$

subject to the condition $x_i(0) = x_i$, $i = 1, 2$. For this reason, the representation (2.11) of the scattering amplitude can be regarded as a functional sum over all possible nucleon paths in the scattering process.

However, the functional integrals (2.14) cannot be integrated exactly and an approximate method must be developed. The simplest possibility is to eliminate $\nu_i(\xi)$ from the argument of the $J_i(k, p_i, q_i|\nu_i)$ function, i.e., we set $\nu_i(\xi) = 0$ in (2.16) for the transition current, and obtain

$$J_i(k, p_i, q_i|\nu_i) = \left[\frac{1}{2p_i k + \mathrm{i}\epsilon} - \frac{1}{2q_i k - \mathrm{i}\epsilon}\right], \tag{2.18}$$

which corresponds to the classical current of a nucleon moving with momentum p for $\xi > 0$ and momentum q for $\xi < 0$.

Note however that the approximation $\nu = 0$ is certainly false for proper time s of the particle near rezo, when the classical trajectory of the particle changes direction. In the language of Feynman diagrams, this corresponds to neglecting the quadratic dependence on k_i in the nucleon propagators, i.e.,

$$\left[m^2 - \left(p - \sum_{i=1}^{n} k_i\right)^2\right]^{-1} \to \left[2p\sum_{i=1}^{n} k_i\right]^{-1}, \tag{2.19}$$

which can lead to the appearance of divergences of integrals with respect to d^4k at the upper limit. As is well known, this approximation, (2.19), can be used to study the infrared asymptotic behavior in quantum electrodynamics [11,20,21]. However, it has not been proved in the region of high energies [11–13].

Therefore, we shall use an approximate method of calculating integrals with respect to $\nu_i(\xi)$ which enables one to retain the quadratic dependence of the nucleon propagators on the momenta k_i. This method is based on the following expansion formula [11,14,22]:

$$\overline{\exp\left(g^2\Pi[\nu]\right)} = \int [\delta^4\nu] \exp(g^2\Pi[\nu]) \tag{2.20}$$
$$= \exp(g^2\overline{\Pi[\nu]})\left[1 + \sum_{n=2}^{\infty} \frac{(g^2)^n}{n!}\overline{(\Pi - \overline{\Pi})^n}\right],$$

where $\overline{\Pi[\nu]} = \int[\delta^4\nu]|\Pi[\nu]$.

Applying the modified expansion formula (2.20) exposed in detail in [28] in our case, we consider the leading term $(n = 0)$ and the following correction term $(n = 1)$. When $n = 0$ the leading term has the form

$$S_\lambda^{(n=0)\text{scalar}} = \overline{\exp\left(\mathrm{i}\lambda g^2\Pi[\nu]\right)} = \int [\delta^4\nu] \exp(\mathrm{i}\lambda g^2\Pi[\nu])$$
$$\approx \exp\left(\mathrm{i}\lambda g^2 \int [\delta^4\nu]\Pi[\nu]\right), \tag{2.21}$$

where

$$\overline{\Pi[\nu]}\Big|_{\nu=0} = \frac{1}{(2\pi)^4}\int \mathrm{d}^4k D(k)\exp(-\mathrm{i}kx)$$
$$\times \int_{-\infty}^{\infty} \mathrm{d}\xi\mathrm{d}\tau \exp\left(2\mathrm{i}k\left[\frac{\xi a_1(\xi)}{\sqrt{s}} - \frac{\tau a_2(\tau)}{\sqrt{s}}\right]\right)$$
$$\times \exp\left[\mathrm{i}\frac{k^2}{\sqrt{s}}(|\xi| + |\tau|)\right]. \tag{2.22}$$

In (2.22), we have made the change of variables $\xi,\tau\to\xi/(s^{1/2}),\tau/(\tau^{1/2})$. When $n=1$ the correction term has the following form:

$$S_\lambda^{(n=1)\text{scalar}}=\exp(\mathrm{i}\lambda g^2\overline{\Pi[\nu]})\times\exp\left[1+\frac{\mathrm{i}\lambda^2g^4}{4}\left(\int\mathrm{d}\eta\sum_{i=1,2}\left(\frac{\delta\overline{\Pi[\nu]}}{\delta\nu_i(\eta)}\right)^2\right)\right]\Bigg|_{\nu=0}. \tag{2.23}$$

Using (2.22) we have

$$\begin{aligned}&\frac{\mathrm{i}\lambda^2g^4}{4}\int\mathrm{d}\eta\left[\left(\frac{\delta\overline{\Pi[\nu]}}{\delta\nu_1(\eta)}\right)^2+\left(\frac{\delta\overline{\Pi[\nu]}}{\delta\nu_2(\eta)}\right)^2\right]\\&=\frac{\mathrm{i}\lambda^2g^4}{(2\pi)^8}\int\mathrm{d}^4k_1\mathrm{d}^4k_2\mathrm{e}^{-\mathrm{i}x(k_1+k_2)}D(k_1)D(k_2)(k_1k_2)\\&\times\int_{-\infty}^{\infty}\mathrm{d}\xi_1\mathrm{d}\tau_1\mathrm{d}\xi_2\mathrm{d}\tau_2\exp\left\{2\mathrm{i}k_1\left[\xi_1\frac{a_1(\xi_1)}{\sqrt{s}}-\tau_1\frac{a_2(\tau_1)}{\sqrt{s}}\right]\right.\\&\times\left.\left[\mathrm{i}\frac{k_1^2}{\sqrt{s}}(|\xi_1|+|\tau_1|)\right]\right\}\\&\times\exp\left\{2\mathrm{i}k_2\left[\xi_2\frac{a_1(\xi_2)}{\sqrt{s}}-\tau_1\frac{a_2(\tau_2)}{\sqrt{s}}\right]\left[\mathrm{i}\frac{k_2^2}{\sqrt{s}}(|\xi_2|+|\tau_2|)\right]\right\}\\&\times\frac{1}{\sqrt{s}}[\Phi(\xi_1,\xi_2)+\Phi(\tau_1,\tau_2)],\end{aligned}\tag{2.24}$$

where

$$\begin{aligned}\Phi(\xi_1,\xi_2)&=\vartheta(\xi_1,\xi_2)[|\xi_1|\vartheta(|\xi_2|-|\xi_1|)+|\xi_2|\vartheta(|\xi_1|-|\xi_2|)],\\\Phi(\tau_1,\tau_2)&=\vartheta(\tau_1,\tau_2)[|\tau_1|\vartheta(|\tau_2|-|\tau_1|)+|\tau_2|\vartheta(|\tau_1|-|\tau_2|)].\end{aligned}\tag{2.25}$$

In this approximation the nucleon propagator functions in (2.21)–(2.25) do not contain terms of type k_ik_j, where k_i and k_j belong to different mesons interacting with the nucleons. This means that in the nucleon propagators we can neglect the terms of the form $\sum_{i\neq j}k_ik_j$ compared with $2p\sum_i k_i$, i.e., we can make the substitution

$$\left[m^2-\left(p-\sum_{i=1}^n k_i\right)^2\right]^{-1}\to\left[2p\sum_{i=1}^n k_i-\sum_{i=1}^n k_i^2\right]^{-1}. \tag{2.26}$$

This approximation, $k_ik_j=0$, which is called the straight-line path approximation, corresponds to the approximate calculation of the Feynman path integrals [9–14, 28, 16, 17] in (2.11) and (2.14) in accordance with the rule (2.26). The formulation of the straight-line path approximation made it possible to put forward a clear physical concept, in accordance with which high-energy particles move along Feynman paths that are most nearly rectilinear.

The validity of the given approximation of (2.26) in the region of high energies s for given momentum transfers t can be studied within the framework of perturbation theory. In particular, one can show that neglecting the terms $k_ik_j=0$ the denominators of the nucleon propagator functions in the case of ordinary ladder diagrams obtained by iteration of the single-meson exchange diagram does not affect the asymptotic behavior at high energies, which, when mesons are exchanged, has the form $\ln s/s^{n-1}$. The validity of this approximation, (2.26), has also been proved for the larger class of diagrams with interacting meson lines [11]. In addition, it should be noted that the eikonal approximation in the potential scattering also reduces to a modification of the propagator (which is nonrelativistic in this case), a modification determined [25] by (2.19) and (2.26).

We shall seek the asymptotic behavior of the functional integral S_λ at large $s=(p_1+p_2)^2$ and fixed momentum transfers $t=(p_1-q_1)^2$. For this, we go over to the center-of-mass system and take the z axis along the moment of the incident particles. Then

$$\begin{aligned}p_{1,2}&=\left\{\frac{\sqrt{s}}{2},0,0,\pm\frac{\sqrt{s-4m^2}}{2}\right\};\\q_{1,2}&=\left\{\frac{\sqrt{s}}{2},\pm\triangle_\perp\sqrt{1+\frac{t}{s-4m^2}}\right.\\&\quad\left.\pm\frac{\sqrt{s-4m^2}}{2}\left(1+\frac{2t}{s-4m^2}\right)\right\},\\\triangle_\perp^2&=-t.\end{aligned}\tag{2.27}$$

Substituting (2.27) into (2.14), we obtain

$$\begin{aligned}a_{1,2}(\xi)&=\frac{1}{\sqrt{s}}[p_{1,2}\theta(\xi)+q_{1,2}\theta(-\xi)]\\&=\frac{1}{2}[\theta(\xi)+\theta(-\xi)]\pm\left(\frac{\Delta_\perp}{\sqrt{s}}\sqrt{1+\frac{t}{s-4m^2}}\right)\theta(-\xi)\\&\pm\frac{\sqrt{s-4m^2}}{\sqrt{s}}\left(1+\frac{t}{s-4m^2}\right).\end{aligned}\tag{2.28}$$

In the limit $s\to\infty$ for fixed t and keeping the terms to order $O(1/s)$, we found

$$\begin{aligned}\frac{a_1(\xi)}{\sqrt{s}}&\approx\frac{1}{2}n^++\frac{\triangle_\perp}{\sqrt{s}}\vartheta(-\xi)+O\left(\frac{1}{s}\right),\\\frac{a_2(\xi)}{\sqrt{s}}&\approx\frac{1}{2}n^--\frac{\triangle_\perp}{\sqrt{s}}\vartheta(-\xi)+O\left(\frac{1}{s}\right),\\n^\pm&=\{1,0,0,\pm1\}.\end{aligned}\tag{2.29}$$

We now find the asymptotic behavior of the expressions (2.22) and (2.24) as $s\to\infty$ and fixed t. Using (2.29), we obtain an asymptotic expression for (2.22) and (2.24). Namely

$$\begin{aligned}\overline{\Pi[\nu]}&=\frac{1}{(2\pi)^6s}\int\mathrm{d}^4k\mathrm{e}^{-\mathrm{i}kx}D(k)\int_{-\infty}^{\infty}\mathrm{d}\xi\mathrm{d}\tau\mathrm{e}^{\mathrm{i}(k_-\xi-k_+\tau)}\\&\times\left\{1-2\mathrm{i}\frac{k_\perp\triangle_\perp}{\sqrt{s}}[\xi\vartheta(-\xi)+\tau\vartheta(-\tau)]+\frac{\mathrm{i}k^2}{\sqrt{s}(|\xi|+|\tau|)}\right\}\\&\approx-\frac{1}{8\pi^2s}\int\frac{\mathrm{d}^2k_\perp}{k_\perp^2+\mu^2}\mathrm{e}^{\mathrm{i}k_\perp x_\perp}\\&+\frac{\mathrm{i}\triangle_\perp}{s\sqrt{s}8\pi^2}[x_+\vartheta(-x_+)-x_-\vartheta(x_-)]\end{aligned}$$

$$
\begin{aligned}
&\times \int \mathrm{d}^2k_\perp \mathrm{e}^{\mathrm{i}k_\perp x_\perp} \frac{k_\perp}{k_\perp^2+\mu^2} \\
&+ \frac{\mathrm{i}}{16\pi^2 s\sqrt{s}}(|x_+|+|x_-|)\int \frac{\mathrm{d}^2k_\perp}{k_\perp^2+\mu^2}\mathrm{e}^{\mathrm{i}k_\perp x_\perp} \\
&= -\frac{1}{4\pi s}K_0(\mu|x_\perp|) \\
&- \frac{\mu}{4\pi s\sqrt{s}}\frac{\triangle_\perp x_\perp}{|x_\perp|}[x_+\vartheta(-x_+)-x_-\vartheta(x_-)]K_1(\mu|x_\perp|) \\
&- \frac{\mathrm{i}\mu^2}{8\pi s\sqrt{s}}(|x_+|+|x_-|)K_0(\mu|x_\perp|), \qquad (2.30)
\end{aligned}
$$

where $x_\pm = x_0 \pm x_z$, the light cone coordinates, $k_\pm^{(i)} = k_0^{(i)} \pm k_z^{(i)}$, $i=1,2$ and μ is the mass of the changed particle, which must be introduced as an infrared regulator. The final expression is

$$
\begin{aligned}
&\frac{\mathrm{i}\lambda^2 g^4}{4}\int \mathrm{d}\eta \left[\left(\frac{\delta\overline{\Pi[\nu]}}{\delta\nu_1(\eta)}\right)^2 + \left(\frac{\delta\overline{\Pi[\nu]}}{\delta\nu_2(\eta)}\right)^2\right] \\
&\approx -\frac{\mathrm{i}\lambda^2 g^4}{(2\pi)^8 s^2\sqrt{s}}\int \mathrm{d}^4k_1\mathrm{d}^4k_2 D(k_1)D(k_2) \\
&\times \exp[-\mathrm{i}x(k_1+k_2)](k_1k_2) \\
&\times \int_{-\infty}^{\infty}\mathrm{d}\xi_1\mathrm{d}\tau_1\mathrm{e}^{\mathrm{i}(k_-^{(1)}\xi_1-k_+^{(1)}\tau_1)}\int_{-\infty}^{\infty}\mathrm{d}\xi_2\mathrm{d}\tau_2\mathrm{e}^{\mathrm{i}(k_-^{(2)}\xi_2-k_+^{(2)}\tau_2)} \\
&\times [\Phi(\xi_1,\xi_2)+\Phi(\tau_1,\tau_2)] \\
&= -\frac{\mathrm{i}\lambda^2 g^4\mu^2}{32\pi^2 s^2\sqrt{s}}(|x_+|+|x_-|)K_1^2(\mu|x_\perp|); \qquad (2.31)
\end{aligned}
$$

here we have assumed $|x_\perp| \neq 0$, which ensures that all the integrals converge. The functions $K_0(\mu|x_\perp|)$ and K_1 $(\mu|x_\perp|)$ are MacDonald functions of the zeroth and first orders and are determined by the expressions

$$
\begin{aligned}
K_0(\mu|x_\perp|) &= \frac{1}{2\pi}\int \mathrm{d}^2k_\perp \frac{\exp(\mathrm{i}k_\perp x_\perp)}{k_\perp^2+\mu^2}, \\
K_1(\mu|x_\perp|) &= -\frac{\partial K_0(\mu|x_\perp|)}{\partial(\mu|x_\perp|)}. \qquad (2.32)
\end{aligned}
$$

We now substitute (2.30) and (2.31) into (2.24) and obtain for the correction term $S_\lambda^{(n=1)}$ the desired expression:

$$
\begin{aligned}
S_\lambda^{(n=1)} \approx &\exp\left[-\frac{\mathrm{i}g^2\lambda}{4\pi s}K_0(\mu|x_\perp|)\right] \\
&\times \left\{1 - \frac{\mathrm{i}g^2\lambda\mu}{4\pi s\sqrt{s}}\frac{\triangle_\perp x_\perp}{|x_\perp|}\right. \\
&\times [x_+\vartheta(-x_+)-x_-\vartheta(x_-)]K_1(\mu|x_\perp|) \\
&+ \frac{g^2\lambda\mu^2}{8\pi s\sqrt{s}}(|x_+|+|x_-|)K_0(\mu|x_\perp|) \\
&\left.- \frac{\mathrm{i}g^4\lambda^2\mu^2}{32\pi^2 s^2\sqrt{s}}(|x_+|+|x_-|)K_1^2(\mu|x_\perp|)\right\}. \qquad (2.33)
\end{aligned}
$$

In this expression, (2.33), the factor in front of the braces corresponds to the leading eikonal behavior of the scattering amplitude, while the terms in the braces determine the correction of relative magnitude $1/(s^{1/2})$.

As is well known from the investigation of the scattering amplitude in the Feynman diagrammatic technique, the high-energy asymptotic behavior can contain only logarithms and integral powers of s. A similar effect is observed here, since integration of the expression (2.33) for S_λ in accordance with (2.14) leads to the vanishing of the coefficients for half-integral powers of s. Nevertheless, allowance for the terms that contain the half-integral powers of s is needed for the calculations of the next corrections in the scattering amplitude. It is interesting to note the appearance in the correction terms of a dependence on x_0 and x_z ($x_\pm = x_0 \pm x_z$), i.e., the appearance of the so-called retardation effects, which are absent in the principal asymptotic term.

Making similar calculations, we can show that all the following terms of the expansion (2.20) decrease sufficiently rapidly compared with those we have written down. However, it must be emphasized that this by no means proves the validity of the eikonal representation for the scattering amplitude in the given framework. The coefficient functions in the asymptotic expansion, which are expressed in terms of MacDonald functions, are singular at short distances and this singularity becomes stronger in an increasing rate with the decrease of the corresponding terms at large s. Therefore, integration of S_λ in accordance with (2.14) in the determination of the scattering amplitude may lead to the appearance of terms that violate the eikonal series in the higher order in g^2. The possible appearance of such terms in individual orders of perturbation theory in models of type φ^3 was pointed out in [23,24,11]. Investigating the structure of the non-eikonal contributions to the two-nucleon scattering amplitude shows that the sum of all ladder diagrams of the eighth order in the scalar model contains terms that are absent in the orthodox eikonal equation and vanish in the limit $(\mu/m) \to 0$, where μ and m are meson and nucleon masses. These terms correspond to the contributions to the effective quasipotential resulting from the exchange of nucleon–antinucleon pairs [28].

To conclude this section we consider the asymptotic behavior of the elastic-scattering amplitude of two scalar nucleons (2.14) in the ultra-high-energy limit $s \to \infty$, $t/s \to 0$. In this case the phase function of the leading eikonal behavior $\chi(b,s) = -g^2/(4\pi s)K_0(\mu|x_\perp|)$ following from (2.33) does not depend on x_+ and x_-. Performing the integration $\mathrm{d}x_+$, $\mathrm{d}x_-$ and $\mathrm{d}\lambda$ for the scattering amplitude in the center-of-mass (c.m.s) system[2] we obtain the following eikonal form:

$$
T(s,t) = -2\mathrm{i}s\int \mathrm{d}^2x_\perp \mathrm{e}^{\mathrm{i}\Delta_\perp x_\perp}(\mathrm{e}^{-\mathrm{i}\chi(x_\perp s)}-1), \qquad (2.34)
$$

where $x_\perp$ is a two-dimensional vector perpendicular to the nucleon-collision direction (the impact parameter), and

[2] The amplitude $T(s,t)$ is normalized in the c.m.s. by the relation

$$
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{|T(s,t)|^2}{64\pi^2 s}, \quad \sigma_t = \frac{1}{2p\sqrt{s}}\mathrm{Im}T(s,t=0)
$$

the eikonal phase function $\chi(x_\perp s)$ by scalar meson exchange decreases with energy:

$$\chi(x_\perp, s) = \frac{g^2}{4\pi s} K_0(\mu|x_\perp|). \tag{2.35}$$

For a similar calculation it has been shown that the exchange term ($p_1 \leftrightarrow p_2$) is one order ($1/s$) smaller and so can be dropped in (2.33). The amplitude is in an eikonal form. The case of interaction of nucleons with vector mesons, and the graviton, can be treated in a similar manner.

3 Corrections to the eikonal equations in quantum gravity

In the framework of standard field theory for the high-energy scattering the different methods have been developed to investigate the asymptotic behavior of individual Feynman diagrams and their subsequent summation. The calculations of eikonal diagrams in the case of gravity run in a similar way as the analogous calculations in QED. The eikonal captures the leading behavior of each order in perturbation theory, but the sum of leading terms is subdominant to the terms neglected by this approximation. The reliability of the eikonal amplitude for gravity is uncertain. One approach which has probed the first of these features with some success is that based on reggeized string exchange amplitudes with subsequent reduction to the gravitational eikonal limit including the leading order corrections [2,26,27]. In this paper we follow a somewhat different approach based on a representation of the solutions of the exact equation of the theory in the form of a functional integral. By this approach we obtain the closed relativistically invariant crossing symmetry expressions for the two-nucleon elastic-scattering amplitudes [9], which may be regarded as sum over all trajectories of the colliding nucleon and are helpful to investigate the asymptotical behavior of scattering amplitudes in different kinematics at low to high energies.

We consider the scalar nucleons $\varphi(x)$ interacting with the gravitational field $g_{\mu\nu}(x)$, where the interaction Lagrangian is of the form

$$L(x) = \frac{\sqrt{-g}}{2}[g^{\mu\nu}(x)\partial_\mu\varphi(x)\partial_\nu\varphi(x) - m^2\varphi^2(x)] + L_{\text{grav.}}(x), \tag{3.1}$$

where $g = \det g_{\mu\nu}(x) = (-g)^{1/2} g^{\mu\nu}(x)$. For the single-particle Green function in the gravitational field $g^{\mu\nu}(x)$ in the harmonic coordinates defined by the condition $\partial_\mu \tilde{g}^{\mu\nu}(x) = 0$, we have the following equation:

$$[\tilde{g}^{\mu\nu}(x)\mathrm{i}\partial_\mu \mathrm{i}\partial_\nu - \sqrt{-g}m^2]G(x,y|g^{\mu\nu}) = \delta^4(x-y), \tag{3.2}$$

whose solution can be written in the form of a functional integral [9]:

$$G(x,y|g^{\mu\nu}) = \mathrm{i}\int_0^\infty \mathrm{d}\tau \mathrm{e}^{-\mathrm{i}m^2\tau} \times C_\nu \int \delta^4\nu \exp\left(-\mathrm{i}\int_0^\tau \mathrm{d}\xi [\tilde{g}^{\mu\nu}(x,\xi)]^{-1}\nu_\mu(\xi)\nu_\nu(\xi) - \mathrm{i}m^2\int_0^\tau [\sqrt{-g(x_\xi)} - 1]\mathrm{d}\xi\right)\delta^4\left(x-y-2\int_0^\tau \nu(\eta)\mathrm{d}\eta\right). \tag{3.3}$$

Equation (3.3) is the exactly closed expression for the scalar-particle Green function in an arbitrary external gravitational field $g^{\mu\nu}(x)$ in the form of a functional integral [9].

In the following we consider the gravitational field in the linear approximation, i.e., we put $g^{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski metric tensor with diagonal $(1,-1,-1,-1)$.

Rewrite (3.3) in the variables $h_{\mu\nu}(x)$ and after dropping the term with an exponent power higher than the first $h_{\mu\nu}(x)$[3], we have a Green function for the single-particle Klein–Gordon equation in a linearized gravitational field:

$$G(x,y|h^{\mu\nu}) = \mathrm{i}\int_0^\infty \mathrm{d}\tau \mathrm{e}^{-\mathrm{i}m^2\tau}\int [\delta^4\nu]_0^\tau \exp\left(\mathrm{i}\kappa\int J_{\mu\nu}(z)h^{\mu\nu}(z)\mathrm{d}z\right) \times \delta^4\left(x-y-2\int_0^\tau \nu(\eta)\mathrm{d}\eta\right), \tag{3.4}$$

where $J_{\mu\nu}(z)$ is the current of the nucleon defined by

$$J_{\mu\nu}(z) = \int_0^{\tau_i} \mathrm{d}\xi(\nu_\mu(\xi)\nu_\nu(\xi)) \times \delta\left(z - x_i + 2p_i\xi + 2\int_0^\xi \nu_i(\eta)\mathrm{d}\eta\right). \tag{3.5}$$

Substituting (3.4) into (2.2) and making analogous calculations as has been done in [9], for the scattering amplitudes we obtain the following expression:

$$T(p_1,p_2;q_1,q_2)^{\text{tensor}} = \kappa^2\int \mathrm{d}^4x \mathrm{e}^{\mathrm{i}(p_1-q_1)x}\varDelta(x;p_1,p_2;q_1,q_2) \times \int_0^1 \mathrm{d}\lambda S_\lambda + (p_1 \leftrightarrow p_2), \tag{3.6}$$

where

$$S_\lambda^{\text{tensor}} = \int \prod_{i=1}^2 [\delta^4\nu_i]_{-\infty}^{\infty} \exp\{\mathrm{i}\kappa^2\lambda\varPi[\nu]\},$$

[3] The Lagrangian (3.1) in the linear approximation to $h^{\mu\nu}(x)$ has the form $L(x) = L_{0,\varphi}(x) + L_{0,\text{grav.}}(x) + L_{\text{int}}(x)$, where

$$L_0(x) = \frac{1}{2}[\partial^\mu\varphi(x)\partial_\mu\varphi(x) - m^2\varphi^2(x)],$$
$$L_{\text{int}}(x) = -\frac{\kappa}{2}h^{\mu\nu}(x)T_{\mu\nu}(x),$$
$$T_{\mu\nu}(x) = \partial_\mu\varphi(x)\partial_\nu\varphi(x) - \frac{1}{2}\eta_{\mu\nu}[\partial^\sigma\varphi(x)\partial_\sigma\varphi(x) - m^2\varphi^2(x)],$$

where $T_{\mu\nu}(x)$ is the energy momentum tensor of the scalar field. The coupling constant κ is related to Newton's constant of gravitation G by $\kappa^2 = 16\pi G$

$$\Pi[\nu] = \int J_1 D J_2 \tag{3.7}$$

$$\Delta(x;p_1,p_2;q_1,q_2) = \int \mathrm{d}^4 k D^{\mu\nu\rho\sigma}(k)\mathrm{e}^{\mathrm{i}kx} \times [k+p_1+q_1]_\mu [k+p_1+q_1]_\nu \times [-k+p_2+q_2]_\rho [-k+p_2+q_2]_\sigma . \tag{3.8}$$

The quantity $J_i^{\mu\nu}(k;p_i,q_i|\nu_i)$ in (3.7) is a conserving transition current given by

$$J_i^{\mu\nu}(k;p_i,q_i|\nu) = 4\int_{-\infty}^{\infty} \mathrm{d}\xi [a_i(\xi)+\nu(\xi)]^\mu [a_i(\xi)+\nu(\xi)]^\nu \times \exp\left(2\mathrm{i}k\left[\xi_i a_i(\xi) + \int_0^\xi \nu_i(\eta)\mathrm{d}\eta\right]\right), \tag{3.9}$$

and $D_{\alpha\beta\gamma\delta}(x)$ is the causal Green function

$$D_{\alpha\beta\gamma\delta}(x) = \omega_{\alpha\beta,\gamma\delta}\frac{\mathrm{i}}{(2\pi)^4}\int \frac{\mathrm{e}^{\mathrm{i}kx}}{k^2-\mu^2+\mathrm{i}\epsilon}\mathrm{d}^4k,$$
$$\omega_{\alpha\beta,\gamma\delta} = (\eta_{\alpha\gamma}\eta_{\beta\delta} + \eta_{\alpha\delta}\eta_{\beta\gamma} - \eta_{\alpha\beta}\eta_{\gamma\delta}).$$

The leading term ($n = 0$) and the following correction term ($n = 1$) in the case of quantum gravity can be constructed in a way similar as in the scalar model,

$$S_\lambda^{(n=0)\mathrm{tensor}} = \int [\delta^4\nu]\exp(\mathrm{i}\lambda g^2 \Pi[\nu]) \approx \exp\left(\mathrm{i}\lambda\kappa^2\int[\delta^4\nu]\Pi[\nu]\right), \tag{3.10}$$

where

$$\overline{\Pi[\nu]}\Big|_{\nu=0} = \frac{1}{(2\pi)^4}\int \mathrm{d}^4k\mathrm{e}^{-\mathrm{i}kx}\int_{-\infty}^{\infty}\mathrm{d}\xi\mathrm{d}\tau a_1^\mu(\xi)a_1^\nu(\xi) \times D_{\mu\nu\sigma\varrho}(k)a_2^\sigma(\tau)a_2^\varrho(\tau)\exp\left(2\mathrm{i}k\left[\frac{\xi a_1(\xi)}{\sqrt{s}} - \frac{\tau a_2(\tau)}{\sqrt{s}}\right]\right) \times \exp\left[\mathrm{i}\frac{k^2}{\sqrt{s}}(|\xi|+|\tau|)\right], \tag{3.11}$$

and

$$S_\lambda^{(n=1)\mathrm{tensor}} = \exp(\mathrm{i}\lambda\kappa^2\overline{\Pi[\nu]}) \times \exp\left[1+\frac{\mathrm{i}\lambda^2\kappa^4}{4}\left(\int\mathrm{d}\eta\sum_{i=1,2}\left(\frac{\delta\overline{\Pi[\nu]}}{\delta\nu_i(\eta)}\right)^2\right)\right]\Bigg|_{\nu=0}. \tag{3.12}$$

Using (2.27) and (2.29), we obtain an asymptotic expression for (3.11) and (3.12), namely,

$$\overline{\Pi[\nu]} = \frac{1}{(2\pi)^6 s}\int\mathrm{d}^4k\mathrm{e}^{-\mathrm{i}kx} \times \int_{-\infty}^{\infty}\mathrm{d}\xi\mathrm{d}\tau\mathrm{e}^{\mathrm{i}(k_-\xi-k_+\tau)}a_1^\mu(\xi)a_1^\nu(\xi)D_{\mu\nu\sigma\varrho}(k)a_2^\sigma(\tau)a_2^\varrho(\tau) \times\left\{1-2\mathrm{i}\frac{k_\perp\triangle_\perp}{\sqrt{s}}[\xi\vartheta(-\xi)+\tau\vartheta(-\tau)] + \frac{\mathrm{i}k^2}{\sqrt{s}(|\xi|+|\tau|)}\right\}$$
$$\approx \frac{s}{4\pi^2}\int\frac{\mathrm{d}^2k_\perp}{k_\perp^2+\mu^2}\exp(\mathrm{i}k_\perp x_\perp) + \frac{\mathrm{i}s\triangle_\perp}{4\pi^2\sqrt{s}}[x_+\vartheta(-x_+) - x_-\vartheta(x_-)] \times\int\mathrm{d}^2k_\perp\exp(\mathrm{i}k_\perp x_\perp)\frac{k_\perp}{k_\perp^2+\mu^2} - \frac{\mathrm{i}s}{8\pi^2\sqrt{s}}(|x_+|+|x_-|)\int\frac{\mathrm{d}^2k_\perp}{k_\perp^2+\mu^2}\exp(\mathrm{i}k_\perp x_\perp)$$
$$= \frac{s}{2\pi}K_0(\mu|x_\perp|) + \frac{s\mu}{2\pi\sqrt{s}}\frac{\triangle_\perp x_\perp}{|x_\perp|}[x_+\vartheta(-x_+) - x_-\vartheta(x_-)]K_1(\mu|x_\perp|) - \frac{\mathrm{i}s\mu^2}{4\pi\sqrt{s}}(|x_+|+|x_-|)K_0(\mu|x_\perp|). \tag{3.13}$$

Then the final expression is

$$\frac{\mathrm{i}\lambda^2\kappa^4}{4}\int\mathrm{d}\eta\left[\left(\frac{\delta\overline{\Pi[\nu]}}{\delta\nu_1(\eta)}\right)^2 + \left(\frac{\delta\overline{\Pi[\nu]}}{\delta\nu_2(\eta)}\right)^2\right]$$
$$\approx -\frac{\mathrm{i}\lambda^2\kappa^4}{(2\pi)^8 s^2\sqrt{s}}\int\mathrm{d}^4k_1\mathrm{d}^4k_2\exp[-\mathrm{i}x(k_1+k_2)](k_1k_2) \times\int_{-\infty}^{\infty}\mathrm{d}\xi_1\mathrm{d}\tau_1\mathrm{e}^{\mathrm{i}(k_-^{(1)}\xi_1-k_+^{(1)}\tau_1)}\int_{-\infty}^{\infty}\mathrm{d}\xi_2\mathrm{d}\tau_2\mathrm{e}^{\mathrm{i}(k_-^{(2)}\xi_2-k_+^{(2)}\tau_2)}$$
$$\times[\Phi(\xi_1,\xi_2)+\Phi(\tau_1,\tau_2)] \times a_1^\mu(\xi_1)a_1^\nu(\xi_1)D_{\mu\nu\sigma\varrho}(k_1)a_2^\sigma(\tau_1)a_2^\varrho(\tau_1)a_1^\rho(\xi_2)a_1^\lambda(\xi_2) \times D_{\rho\lambda\eta\omega}(k_2)a_2^\eta(\tau_2)a_2^\omega(\tau_2)[\Phi(\xi_1,\xi_2)+\Phi(\tau_1,\tau_2)]$$
$$= \frac{\mathrm{i}\lambda^2\kappa^4 s^2\mu^2}{8\pi^2\sqrt{s}}(|x_+|+|x_-|)K_1^2(\mu|x_\perp|). \tag{3.14}$$

As in the preceding section we have assumed $|x_\perp| \neq 0$, which ensures that all the integrals converge. We now substitute (3.13) and (3.14) into (3.12) and obtain for $S_\lambda^{(n=1)\mathrm{tensor}}$ the desired expression,

$$S_\lambda^{(n=1)\mathrm{tensor}} \approx \exp\left[\frac{\mathrm{i}\kappa^2 s\lambda}{2\pi}K_0(\mu|x_\perp|)\right] \times\left\{1+\frac{\mathrm{i}\kappa^2 s\lambda\mu}{2\pi\sqrt{s}}\frac{\triangle_\perp x_\perp}{|x_\perp|} \times[x_+\vartheta(-x_+)-x_-\vartheta(x_-)]K_1(\mu|x_\perp|) - \frac{\kappa^2 s\lambda\mu^2}{4\pi\sqrt{s}}(|x_+|+|x_-|)K_0(\mu|x_\perp|) + \frac{\mathrm{i}\kappa^4 s^2\lambda^2\mu^2}{8\pi^2\sqrt{s}}(|x_+|+|x_-|)K_1^2(\mu|x_\perp|)\right\}. \tag{3.15}$$

It is important to note that in contrast to the scalar model the corresponding correction terms in quantum gravity increase with the energy. Using (3.14) and the phase function of the leading eikonal behavior following from (3.15), after integration over $\mathrm{d}x_+$, $\mathrm{d}x_-$ and $\mathrm{d}\lambda$ for the scattering amplitude in the high-energy limit $s \gg M_{\mathrm{PL}}^2 \gg t$, we obtain the following eikonal form:

$$T(s,t)^{\mathrm{tensor}} = -2\mathrm{i}s\int\mathrm{d}^2x_\perp\mathrm{e}^{\mathrm{i}\Delta_\perp x_\perp}(\mathrm{e}^{\mathrm{i}\chi(|x_\perp|s)}-1), \tag{3.16}$$

where the eikonal phase function $\chi(x_\perp s)$ by graviton exchange increases with energy as

$$\chi(x_\perp s) = \frac{\kappa^2 s}{2\pi} K_0(\mu|\vec{x}_\perp|), \tag{3.17}$$

and in the model with vector mesons ($L_{\text{int}} = -g\varphi^\star \mathrm{i}\partial_\sigma \varphi A^\sigma + g^2 A_\sigma A^\sigma \varphi^\star \varphi$), the eikonal phase function is

$$\chi(x_\perp) = \frac{g^2}{2\pi} K_0(\mu|\vec{x}_\perp). \tag{3.18}$$

It should be noted that the eikonal phases given by (2.34), (3.18) and (3.17) correspond to a Yukawa potential between the interacting nucleons; according to the spin of the exchange field in the scalar case this potential decreases with energy $V(s,|x_\perp|) = -(g^2/8\pi s)(\mathrm{e}^{-\mu|x_\perp|}/|x_\perp|)$ and is independent of energy in the vector model $V(s, |x_\perp|) = -(g^2/4\pi)(\mathrm{e}^{-\mu|x_\perp|}/|x_\perp|)$. In the case of graviton exchange the Yukawa potential $V(s,|x_\perp|) = (\kappa^2 s/2\pi)(\mathrm{e}^{-\mu|x_\perp|}/|x_\perp|)$ increases with energy. Comparison of these potentials has made it possible to draw the following conclusions: in the model with scalar exchange, the total cross section σ_t decreases as $1/s$, and only the Born term predominates in the entire eikonal equation; the vector model leads to a total cross section σ_t tending to a constant value as $s \to \infty$, $t/s \to 0$. In both cases, the eikonal phases are purely real and consequently the influence of inelastic scattering is disregarded in this approximation, $\sigma^{\text{in}} = 0$. In the case of graviton exchange the Froissart limit is violated. A similar result is also obtained in [6] with the eikonal series for reggeized graviton exchange.

We may mention that in the framework of the quasipotential approach [29–31] in quantum field theory there is a rigorous justification of the eikonal representation on the basis of the assumption of a smooth local quasipotential. In the determination of non-leading terms just considered we have a singular interaction which, when radiative effects are ignored, leads to a singular quasipotential of the Yukawa type which requires special care.

4 Conclusions

In the framework of functional integration using the straight-line path approximation in quantum gravity we obtained the first-order correction terms to the leading eikonal behavior of the Planck energy scattering amplitude. We have also shown that the allowance for these terms leads to the appearance of retardation effects, which are absent in the principal asymptotic term. It is important to note that the singular character of the correction terms at short distances may ultimately lead to the appearance of non-eikonal contributions to the scattering amplitudes. The straight-line paths approximation used in this work corresponds to a physical picture in which colliding high-energy nucleons in the process of interaction receive a small recoil connected with the emission of "soft" mesons or gravitons and retain their individuality. The calculation of non-leading terms to leading eikonal behavior of Planck energy scattering can be realized by means of the quasipotential method which provides a consistent justification of the eikonal representation of the scattering amplitude with a smooth local quasipotential. This problem requires some further study.

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